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Abstract

The Riemann equation $u_t + uu_x = 0$, which describes a one-dimensional accelerationless perfect fluid, possesses solutions that typically develop shocks in a finite time. This equation is $\mathcal{PT}$ symmetric. A one-parameter $\mathcal{PT}$-invariant complex deformation of this equation, $u_t - iu(iu_x)^\epsilon = 0$ (ϵ real), is solved exactly using the method of characteristic strips, and it is shown that for real initial conditions, shocks cannot develop unless ϵ is an odd integer. When ϵ is an odd integer, the shock-formation time is calculated explicitly.

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1. Introduction

The concept of $\mathcal{PT}$-symmetric quantum mechanics was introduced in 1998 [1]. Since then, there have been many studies of the properties of such quantum theories (see, for example [2–6]). In an effort to understand the underlying mathematical structure of $\mathcal{PT}$-symmetric quantum mechanics, there have recently been a number of studies of classical differential equations that are symmetric under $\mathcal{PT}$ reflection. These studies include an examination of the complex solutions of the differential equations for the classical–mechanical systems associated with $\mathcal{PT}$-symmetric quantum–mechanical systems [2, 7–9]. Additional studies have been made of various $\mathcal{PT}$-symmetric classical–mechanical systems such as the $\mathcal{PT}$-symmetric pendulum [10], the $\mathcal{PT}$-symmetric Lotka–Volterra equation [11, 12] and the $\mathcal{PT}$-symmetric Euler equations that describe free rigid-body rotation [12].

Many well-known nonlinear wave equations exhibit $\mathcal{PT}$ symmetry. For example, the Korteweg-de Vries and generalized Korteweg-de Vries equations [13], the Camassa–Holm equation [15], the Sine-Gordon and Boussinesq equations [13] are all $\mathcal{PT}$ symmetric. Once it

is known that a differential equation is $\mathcal{PT}$ symmetric, it is possible to introduce a continuous parameter ϵ to deform the equation into the complex domain in such a way as to preserve the $\mathcal{PT}$ symmetry. Such a deformation gives rise to large parametric classes of new equations whose properties exhibit an interesting dependence on ϵ. For example, in recent papers a complex family of nonlinear wave equations obtained by deforming the Korteweg-de Vries equation was examined [16, 17].

In this paper we study a simplified version of the complex class of Korteweg-de Vries equations considered in [16]; namely, the first-order nonlinear wave equations obtained by removing the dispersive third-order-derivative term. This class of equations is the $\mathcal{PT}$-symmetric deformation of the Riemann equation. The virtue of studying this simpler class of equations is that they can be solved *exactly and in closed form* for all values of the deformation parameter ϵ, as we show in this paper. This allows us to determine how the properties of the complex solution depend on ϵ.

The well known and heavily studied Riemann equation

$$u_t + uu_x = 0 \tag{1}$$

describes a one-dimensional incompressible accelerationless fluid. The velocity profile $u(x, t)$ represents the velocity of the fluid at the point x and at time t. For the initial condition

$$u(x, 0) = f(x) \tag{2}$$

the exact solution to (1) can be written in implicit form as

$$u = f(x - ut). \tag{3}$$

The Riemann equation (1) is symmetric under $\mathcal{PT}$ reflection, where $\mathcal{P}$ represents spatial reflection and $\mathcal{T}$ represents time reversal. (Note that under $\mathcal{PT}$ reflection u does not change sign because it is a velocity.) A simple way to deform the Riemann equation into the complex domain in such a way as to preserve its $\mathcal{PT}$ invariance is to introduce the real parameter ϵ as follows:

$$u_t - iu(iu_x)^\epsilon = 0. \tag{4}$$

Note that as in quantum mechanics, we assume that $\mathcal{T}$ is an antilinear operator that has the effect of reversing the sign of the complex number i. For the special value $\epsilon = 1$ this equation reduces to the conventional Riemann equation (1). We will examine the solutions to (4) that arise from the initial condition in (2).

Let us recall how to solve the standard Riemann equation in (1). The Riemann equation is a typical example of a *quasilinear* partial differential equation (one that is linear in the partial derivatives) that can be solved using the method of characteristics. To do so we parametrize the initial condition ({2) in terms of the parameter r

$$t = 0, \qquad x = r, \qquad u = f(r). \tag{5}$$

Next, we construct the system of three coupled ordinary differential equations known as the *characteristic differential equations*

$$\frac{dt}{ds} = 1, \qquad \frac{dx}{ds} = u, \qquad \frac{du}{ds} = 0. \tag{6}$$

In general, when the characteristics for a quasilinear equation are solved subject to the initial conditions imposed at $s = 0$, the resulting solution has the general form $x = x(s, r), t = t(s, r), u = u(s, r)$. Carrying out this procedure for the case of the Riemann equation produces the solution

$$t = s, \qquad x = f(r)s + r, \qquad u = f(r). \tag{7}$$

The last step is to eliminate the variables r and s in favor of the original variables x and t, which yields the solution $u(x, t)$ of (1) given implicitly in (3).

It is well known that for most choices of the initial condition $f(x)$ the solution $u(x, t)$ eventually becomes a multiple-valued function of x. This happens because the characteristic curves in the (x, t) plane for the Riemann equation are the straight lines $x = f(x_0)t + x_0$, as we can see from (7). These lines are in general not all parallel because f is not a constant, and thus some of the characteristics may eventually cross. It is physically unacceptable for the velocity field of a fluid to be multiple valued. To avoid multiple-valued solutions, it is conventional to require that when two characteristic lines cross, a shock develops. A *shock* is a jump discontinuity as a function of x in the fluid profile $u(x, t)$. A solution $u(x, t)$ containing a shock is not differentiable at the position of the shock and it is called a *weak solution* [13].

The purpose of this paper is to show that deforming the Riemann equation into the complex domain has the effect of softening the shock singularities, as was conjectured in [14]. Specifically, we will show in the following section that shocks can only form when ϵ takes odd-integer values. To show this we will solve the deformed Riemann equation (4) exactly using the method of characteristic strips, which is explained in the appendix.

2. Shock formation time for the $\mathcal{PT}$-deformed Riemann equation

To find out whether the $\mathcal{PT}$-deformed Riemann equation (4) develops shocks, we examine the problem quantitatively: we calculate the precise amount of time required for a shock to form. To demonstrate the technique, we begin by calculating the shock-formation time for the standard Riemann equation in (1).

A shock forms as soon as the coordinate transformation in (7) becomes singular. To determine when this happens we calculate the Jacobian of this transformation

$$J = \frac{\partial(t, x)}{\partial(s, r)} = \det \begin{bmatrix} 1 & 0 \\ f(r) & sf'(r) + 1 \end{bmatrix} = 1 + sf'(r). \tag{8}$$

This transformation ceases to be invertible as soon as J vanishes, namely, at the *minimal positive* value of $s = -1/f'(r)$. Thus, the shock-formation time t_{shock} is given by

$$t_{\text{shock}} = \overset{\text{positive min}}{\underset{x \in \mathbb{R}}{}} [-1/f'(x)]. \tag{9}$$

For $\epsilon \neq 1$, (4) is nonlinear, and not quasilinear, and the method of characteristics does not apply. Instead, we shall solve (4) using the method of *characteristic strips*, a generalization of the method of characteristics, which in some cases can be used to solve nonlinear first-order partial differential equations.

The method of characteristic strips is explained briefly in the appendix. Using the notation in the appendix, we rewrite (4) as $F(x, y, u, p, q) = 0$, where $t \equiv y$, $p = u_x$, $q = u_t$, and

$$F = q - iu(ip)^\epsilon. \tag{10}$$

The procedure is to seek a coordinate transformation of the form $x = x(s, r)$ and $t = t(s, r)$ such that the five arguments of F satisfy the five characteristic-strip equations in (A.18), namely,

$$\frac{dx}{ds} = -\epsilon q/p, \qquad \frac{dt}{ds} = 1, \qquad \frac{du}{ds} = (1 - \epsilon)q, \qquad \frac{dp}{ds} = pq/u, \qquad \frac{dq}{ds} = q^2/u. \tag{11}$$

Here we have substituted (10) into (A.18) and also substituted $F = 0$ in various places, after taking the appropriate derivatives. Note that for $\epsilon = 1$, the equations for x, t and u agree with the characteristic differential equations (6) for the Riemann equation (1).

Remarkably, the coupled system of ordinary differential equations (11) admits a closed explicit solution[3]

$$p = G(D - \epsilon s)^{-1/\epsilon}, \qquad q = E(D - \epsilon s)^{-1/\epsilon},$$
$$u = E(D - \epsilon s)^{(\epsilon-1)/\epsilon}, \qquad t = s + H, \qquad x = -\epsilon Es/G + I, \tag{12}$$

where D, E, G, H and I are functions of the parameter r and are yet to be determined from the initial conditions (2).

Next, as explained in the appendix, we verify that (12), which solves the characteristic-strip equations (11), constitutes a solution of the partial differential equation (4). To do so we impose the condition that $F(x, t, u, p, q) = 0$ on the five functions in (12). In fact, this condition is nontrivial, and it translates into the condition

$$(iG)^\epsilon = -i, \tag{13}$$

which determines G. (For irrational values of ϵ the solution to this equation has infinitely many branches, and we choose the branch that tends to -1 as $\epsilon \to 1$. This is the solution that corresponds to the solution of the Riemann equation (1) as $\epsilon \to 1$.)

Next, we rewrite the initial conditions $u(x, 0) = f(x)$ in parametric form. On the initial curve at $s = 0$ we have

$$x = r, \qquad t = 0, \qquad u = f(r). \tag{14}$$

Since $u = u(x, t)$, we have $u_r = p\frac{dx}{dr} + q\frac{dt}{dr}$. Also $\frac{du}{dr} = f'(r)$, $\frac{dx}{dr} = 1$, $\frac{dt}{dr} = 0$. Thus,

$$u_r = p(r) = f'(r). \tag{15}$$

Also, since $F = 0$, we deduce from (10) that

$$q(r) = iu(r)[ip(r)]^\epsilon = if(r)[if'(r)]^\epsilon. \tag{16}$$

We impose the initial conditions by setting $s = 0$ in (12). Then, using (15) and (16), we obtain

$$p(0) = f'(r) = GD^{-1/\epsilon}, \qquad q(0) = if(r)[if'(r)]^\epsilon = ED^{-1/\epsilon},$$
$$u(0) = f(r) = ED^{\frac{\epsilon-1}{\epsilon}}, \qquad t(0) = 0 = H, \qquad x(0) = r = I. \tag{17}$$

From the first two equations in (17) we solve for D and E in terms of $f(r)$, $f'(r)$ and G. Substituting these expressions for D and E in the third equation in (17) gives an equation for G that coincides with (13). Hence, we obtain a self-consistent result. Substituting G into the expressions for D and E gives

$$D = -i[if'(r)]^{-\epsilon}, \qquad E/G = -f(r)[if'(r)]^{\epsilon-1}. \tag{18}$$

From the last two equations in (17) we have

$$I = r, \qquad H = 0, \tag{19}$$

and gathering our results from (12), (18) and (19), we finally obtain

$$t = s, \qquad x = r + \epsilon f(r)[if'(r)]^{\epsilon-1}s, \qquad u = f(r). \tag{20}$$

Surprisingly, even though (4) is nonlinear, we have solved it in *closed form*. (Note that for $\epsilon = 1$ this solution reduces to the solution of the Riemann equation (1).)

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[3] It is not always possible to use the method of characteristic strips to solve a nonlinear partial differential equation because the technical assumption that is made in (A.16) may not hold.

The next step in the application of the method of characteristic strips is to solve for r and s in terms of x and t, and to substitute the expression for r into $u(r)$ to obtain the desired solution $u(x, t)$. This can be done if and only if the Jacobian

$$J = \frac{\partial(t, x)}{\partial(s, r)} = \det \begin{bmatrix} \dfrac{\partial t}{\partial s} & \dfrac{\partial t}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial r} \end{bmatrix} = \frac{\partial x}{\partial r} = 1 + \epsilon s \frac{\mathrm{d}}{\mathrm{d}r} \{ f(r)[\mathrm{i} f'(r)]^{\epsilon-1} \} \qquad (21)$$

does not vanish.

Clearly, for *real* $f(r)$, that is, for real initial conditions, J never vanishes unless ϵ is an *odd integer*. This establishes the principal result of this paper. When ϵ is an odd integer, the Jacobian may vanish, and a shock will form at time t_{shock} given by

$$t_{\text{shock}} = \mathop{\text{positive min}}_{r \in \mathbb{R}} \left(\frac{-1}{\epsilon \frac{\mathrm{d}}{\mathrm{d}r} \{ f(r)[\mathrm{i} f'(r)]^{\epsilon-1} \}} \right). \qquad (22)$$

2.1. Case $\epsilon = 2$

For $\epsilon = 2$ (4) can be reduced to the Riemann equation for a *pure-imaginary* velocity field as follows: consider (4) at $\epsilon = 2$ and multiply it by u. Define $v = u^2/2$. The resulting equation is $v_t = -\mathrm{i} v_x^2$. Take the x derivative of both sides and define $w = \mathrm{i} v_x = \mathrm{i} u u_x$. Clearly, for real $u(x, 0) = f(x)$, $w(x, 0) = \mathrm{i} f(x) f'(x)$ is pure-imaginary and w satisfies the Riemann equation $w_t + 2w w_x = 0$. The Jacobian (21) in this case is $J = 1 + 2\mathrm{i} s \frac{\mathrm{d}}{\mathrm{d}x} [f(x) f'(x)]$; J can never vanish for real $f(x)$, in accordance with our general result.

2.2. Case $\epsilon = 3$

According to our general result, shocks may form at $\epsilon = 3$. In this case (22) reads

$$t_{\text{shock}} = \mathop{\text{positive min}}_{r \in \mathbb{R}} \left(\frac{1}{3 \frac{\mathrm{d}}{\mathrm{d}r} \{ f(r)[f'(r)]^2 \}} \right). \qquad (23)$$

As a concrete example, we take the initial velocity profile to be the Cauchy distribution $f(x) = 1/(x^2 + 1)$. For this case, (23) becomes

$$t_{\text{shock}} = \mathop{\text{positive min}}_{x \in \mathbb{R}} \left[\frac{(1 + x^2)^6}{24x(1 - 4x^2)} \right]. \qquad (24)$$

This function has two maxima and two minima, which are located at $x = \pm 0.216\,621$ and at $x = \pm 0.769\,392$. The lower positive minimum occurs at $x = 0.216\,621$, and at this value of x, it equals $0.311\,791$. Thus, this initial condition evolves into a shock at $t = 0.311\,791$.

Appendix. Concise summary of the method of characteristic strips

In this appendix we give a concise recipe for using the method of characteristic strips to solve nonlinear first-order partial differential equations. This appendix is brief, and we remark that a complete discussion of the method of characteristic strips would be rather long and complicated (see, for example, [18, 19]).

We wish to solve the general first-order partial differential equation

$$F(x, y, u, u_x, u_y) = 0 \qquad (A.1)$$

for the unknown function $u(x, y)$ subject to the initial condition $u(x, 0) = f(x)$. (Thus, $u_x(x, 0) = f'(x)$, while $u_y(x, 0)$ is determined by the equation $F = 0$ at $y = 0$.) To this end we introduce two more unknowns, p and q, and rewrite (A.1) as

$$F(x, y, u, p, q) = 0, \tag{A.2}$$

subject to the constraints

$$u = u(x, y), \qquad p = u_x = p(x, y), \qquad q = u_y = q(x, y). \tag{A.3}$$

In order to solve this equation in a manner similar to the method of characteristics, we seek a coordinate transformation of the form $x = x(s, r)$, $y = y(s, r)$, which will have the effect of reducing (A.1) to a closed system of coupled *ordinary* differential equations in the variable s. The initial condition will be parametrized in terms of r.

We can immediately write down eight differential equations associated with the partial differential equation (A.1):

$$p = u_x, \tag{A.4}$$

$$q = u_y, \tag{A.5}$$

$$p_y = q_x, \tag{A.6}$$

$$\frac{\mathrm{d}u}{\mathrm{d}s} = p\frac{\mathrm{d}x}{\mathrm{d}s} + q\frac{\mathrm{d}y}{\mathrm{d}s}, \tag{A.7}$$

$$\frac{\mathrm{d}p}{\mathrm{d}s} = p_x\frac{\mathrm{d}x}{\mathrm{d}s} + p_y\frac{\mathrm{d}y}{\mathrm{d}s}, \tag{A.8}$$

$$\frac{\mathrm{d}q}{\mathrm{d}s} = q_x\frac{\mathrm{d}x}{\mathrm{d}s} + q_y\frac{\mathrm{d}y}{\mathrm{d}s}, \tag{A.9}$$

$$F_x + F_u p + F_p p_x + F_q q_x = 0, \tag{A.10}$$

$$F_y + F_u q + F_p p_y + F_q q_y = 0. \tag{A.11}$$

It is easy to check that (A.10) and (A.11) lead to $\frac{\mathrm{d}F}{\mathrm{d}s} = 0$, which is consistent with (A.2).

Next, we use (A.6) to rewrite (A.10) as

$$F_x + F_u p + F_p p_x + F_q p_y = 0, \tag{A.12}$$

and similarly, we rewrite (A.11) as

$$F_y + F_u q + F_p q_x + F_q q_y = 0. \tag{A.13}$$

Since our goal is to find expressions for the parametric derivatives $\frac{\mathrm{d}x}{\mathrm{d}s}$, $\frac{\mathrm{d}y}{\mathrm{d}s}$, $\frac{\mathrm{d}u}{\mathrm{d}s}$, $\frac{\mathrm{d}p}{\mathrm{d}s}$ and $\frac{\mathrm{d}q}{\mathrm{d}s}$ in terms of x, y, u, p and q, we must try to eliminate all reference to the higher derivatives p_x, p_y, q_x and q_y. We proceed as follows: using (A.8) and (A.9), we eliminate p_y and q_x as

$$p_y = \left(\frac{\mathrm{d}p}{\mathrm{d}s} - p_x\frac{\mathrm{d}x}{\mathrm{d}s}\right)\bigg/\frac{\mathrm{d}y}{\mathrm{d}s} \qquad \text{and} \qquad q_x = \left(\frac{\mathrm{d}q}{\mathrm{d}s} - q_y\frac{\mathrm{d}y}{\mathrm{d}s}\right)\bigg/\frac{\mathrm{d}x}{\mathrm{d}s}. \tag{A.14}$$

Substituting these expressions into (A.11) and (A.12) we obtain

$$F_x + F_u p + p_x\left(F_p - F_q\frac{\mathrm{d}x}{\mathrm{d}s}\bigg/\frac{\mathrm{d}y}{\mathrm{d}s}\right) + F_q\frac{\mathrm{d}p}{\mathrm{d}s}\bigg/\frac{\mathrm{d}y}{\mathrm{d}s} = 0,$$
$$F_y + F_u q + q_y\left(F_q - F_p\frac{\mathrm{d}y}{\mathrm{d}s}\bigg/\frac{\mathrm{d}x}{\mathrm{d}s}\right) + F_p\frac{\mathrm{d}q}{\mathrm{d}s}\bigg/\frac{\mathrm{d}x}{\mathrm{d}s} = 0. \tag{A.15}$$

It is now clear that the only way to obtain a closed system of five coupled ordinary differential equations is to impose the additional requirement that

$$F_p \frac{\mathrm{d}y}{\mathrm{d}s} = F_q \frac{\mathrm{d}x}{\mathrm{d}s}. \tag{A.16}$$

Upon imposing (A.16), we see that (A.11) and (A.12) assume the particularly simple form

$$F_x + F_u p + F_q \frac{\mathrm{d}p}{\mathrm{d}s} \Big/ \frac{\mathrm{d}y}{\mathrm{d}s} = 0, \qquad F_y + F_u q + F_p \frac{\mathrm{d}q}{\mathrm{d}s} \Big/ \frac{\mathrm{d}x}{\mathrm{d}s} = 0. \tag{A.17}$$

We emphasize that the requirement (A.16) is not necessarily valid, and after solving the partial differential equation (A.1) it is necessary to verify the correctness of (A.16).

Next, we make use of the functional degree of freedom to *reparametrize*; that is, to introduce a new parameter $s' = g(s)$, which is an arbitrary function $g(s)$ of the old parameter s. This freedom allows us to choose $\frac{\mathrm{d}x}{\mathrm{d}s} = F_p$, and from (A.16) it follows that $\frac{\mathrm{d}y}{\mathrm{d}s} = F_q$. In this way we obtain the desired s derivatives as

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}s} &= F_p, & \frac{\mathrm{d}y}{\mathrm{d}s} &= F_q, & \frac{\mathrm{d}u}{\mathrm{d}s} &= p F_p + q F_q, \\
\frac{\mathrm{d}p}{\mathrm{d}s} &= -F_x - F_u p, & \frac{\mathrm{d}q}{\mathrm{d}s} &= -F_y - F_u q.
\end{aligned}
\tag{A.18}
$$

These are the characteristic-strip equations. These are a generalization of the three coupled characteristic ordinary differential equations for quasilinear partial differential equations to the five characteristic-strip equations for fully nonlinear partial differential equations. However, we emphasize that there is a new aspect; namely, that after solving the system (A.18), it is necessary to verify that the assumption made in (A.16) is valid.

The problem of solving the nonlinear equation (A.1) is now converted to solving the coupled system (A.18). Once this is done and (A.16) has been verified, it is then necessary to impose the initial conditions, and this will introduce the dependence on the parameter r. The final step is to eliminate r and s in favor of the original variables x and y, and thereby to obtain the solution $u = u(x, y)$.

Note: after the submission of this paper, a paper by Curtright and Fairlie [20] appeared on the web arXiv which gives a simple and elegant solution to (4). This paper does not address the question of shock formation.

Acknowledgments

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